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ANALYSIS OF THE EFFECT OF TRANSVERSE SHEAR DEFORMATION ON THE STIFFNESS OF SHORT BEAMS

Ilchenko Iryna, *PhD (Candidate of Technical Science), Associate Professor, National Transport University, Associate Professor of Strength of Materials and Mechanical Engineering Department, e-mail: budivmeh@gmail.com, tel. +380442804368, Ukraine, 01010, Kyiv, Omelianovych-Pavlenko str. 1, of. 113*

https://orcid.org/0000-0002-9355-5632

Gorbunovich Iryna, *PhD (Candidate of Technical Science), Associate Professor, National Transport University, Associate Professor of Higher Mathematics Department, e-mail:* irina.gorbunovich@gmail.com*, tel. +380676001177, Ukraine, 01103, Kyiv, Boychuka str.,42, of. 514*

https://orcid.org/0000-0002-6859-0663

Summary. The article examines the deformations of short elastic beams with different cross-sections under concentrated and uniformly distributed loads. The effect of transverse shear deformations on the stiffness of the beams is investigated.

A second-order differential equation is used as a mathematical model of the problem, formed with consideration of pure bending and transverse shear deformations [1]. Boundary conditions are specified for simply supported and cantilever beams.

The boundary value problem is solved analytically and using the finite element method (FEM), allowing control over different levels of accuracy. The maximum deflections of beams with different cross-sections (rectangular, I-shape, circular, and annular) are determined. Timoshenko beam theory is applied to obtain beam deflections [2].

Modern structural design relies on computer-aided design (CAD) systems, which enable the creation of precise structural models and the analysis of their behavior under realistic conditions. CAD software with integrated finite element tools enables the analysis of the behavior and the identification of specific deformations in both individual components and the entire structures [3].

Three-dimensional finite element modeling of beams was performed using the SCAD and LIRA software [4, 5]. Graphs of mesh convergence for the deflections of the I-shaped cantilever beams were plotted. A comparison of analytical and numerical solutions of the boundary value problem demonstrated a high level of accuracy, indicating the efficiency of the applied approaches.

The obtained results have practical significance for structural engineers, particularly in analyzing short beam deformations during the design of beams with different cross-sections. Additionally, these research findings can be incorporated into the educational process for the "Strength of Materials" course.

Keywords: simply supported beam, cantilever beam, thin-walled cross-sections, parametric cross-sections, stiffness of the beam, transverse shear deformations, finite element software, LIRA software, SCAD software.

Introduction. Beams are fundamental structural elements in nearly all engineering structures. The assessment of beam deflections is a common design challenge. While classical beam theory (Euler-Bernoulli) simplifies bending analysis, it neglects transverse shear deformations. The Euler-Bernoulli theory is suitable for analytical calculations of deflections in long and slender beams, but significant errors arise when applied to short beams. Beams with a relative length $L \leq 5h$ (h – beam height) are classified as short beams. The Timoshenko theory provides more accurate results for short beams by accounting for both pure bending and shear deformations [2].

Aims. This study aims to determine the effect of transverse shear deformation on the stiffness of simply supported and cantilever beams with different cross-sections (thin-walled and parametric) and to compare the results of analytical solutions based on the differential equation with the results of three-dimensional finite element analysis obtained using the SCAD and LIRA software packages.

Object of study – simply supported and cantilever beams with thin-walled (I-shape, annular) and parametric cross-sections (rectangular, circular).

Main Study. The differential equation of beam bending (fig. 1), which accounts for both pure bending and transverse shear deformations, can be expressed as follows [1]:

$$\frac{d^2w(x)}{dx^2} = \frac{d^2w_{cl}(x)}{dx^2} + \frac{d^2v(x)}{dx^2} = -\frac{M(x)}{EI} - \mu\frac{q(x)}{GF}, \tag{1}$$

where: $w(x) = w$ – total deflection due to both pure bending and transverse shear deformations;
$w_{cl}(x) = w_{cl}$ – classical deflection due to pure bending deformations;
$v(x) = v$ – deflection due to transverse shear deformation;
EI – flexural rigidity;
GF – shear rigidity;
μ – coefficient accounting for the non-uniform distribution of shear stresses across the section height;
$M(x)$ – bending moment;
$q(x)$ – intensity of uniformly distributed load.

According to [6] the following coefficient μ is accepted: $\mu = 1{,}2$ for the rectangular cross-section, $\mu = 2$ for annular cross-section, $\mu = 32/27$ for circular cross-section. For the I-shaped beam $\mu = F/F_{web}$ (F – the I-beam area, F_{web} – the web area).

As an example, a cantilever beam loaded with a uniformly distributed load with intensity $q(x) = q$ is considered (fig. 1). The equation of the bending moment is expressed as follows: $(x) = qLx - \frac{qL^2}{2} - \frac{qx^2}{2}$.

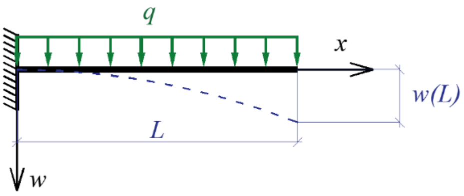


Figure 1 - The cantilever beam under a uniformly distributed load
Рисунок 1 – Консольна балка під дією рівномірно розподіленого навантаження

$$\frac{dw(x)}{dx} = -\left(\frac{qL}{2EI}x^2 - \frac{q}{2EI}L^2x - \frac{q}{6EI}x^3\right) - \mu\frac{q}{GF}x + C_1, \tag{2}$$

$$w(x) = -\left(\frac{qL}{6EI}x^3 - \frac{qL^2x^2}{4EI} - \frac{qx^4}{24EI}\right) - \mu\frac{q}{2GF}x^2 + C_1x + C_2, \tag{3}$$

To establish the boundary conditions, the following equations are formulated according to [1]:

$$\frac{dv(x)}{dx} = \mu\frac{Q(x)}{GF}, \tag{4}$$

where: $Q(x)$ – shear force.

Equations (2) and (3), when subjected to the boundary conditions at $x = 0$, $w(0) = 0$, $\frac{dv(0)}{dx} = \mu\frac{Q(0)}{GF} = \mu\frac{q\cdot L}{GF}$ and $\frac{dw_{cl}(0)}{dx} = 0$ yield the constants $C_1 = \frac{dv(0)}{dx} = \mu\frac{q\cdot L}{GF}$, $C_2 = 0$. Thus:

$$w(x) = -\left(\frac{qL}{6EI}x^3 - \frac{qL^2x^2}{4EI} - \frac{qx^4}{24EI}\right) - \mu\frac{q}{2GF}x^2 + \mu\frac{qL}{GF}x, \tag{5}$$

Equation (5) with the $x = L$ gives the value of maximum deflection of the cantilever beam $w(L) = w_{max}$:

$$w(L) = \frac{qL^4}{8EI} + \mu\frac{qL^2}{2GF}, \tag{6}$$

Solving the differential equation (1) by integration provides the refined expression for deflection, which considers both pure bending and transverse shear deformations. Neglecting the shear deformation terms in equation (1) leads to the classical form:

$$\frac{d^2w_{cl}(x)}{dx^2} = -\frac{M(x)}{EI}, \tag{7}$$

By solving equation (7) via integration, the classical deflection is obtained, considering only pure bending deformation.

For example, for the cantilever beam (Figure 1) under a uniformly distributed load the classical deflection formula takes form of: $w_{cl}(L) = \frac{qL^4}{8EI}$. Based on the Maxwell-Mohr formula, the formulas for the maximum deflections of beams are derived in [2].

The effect of transverse shear deformations on the stiffness of the beams is quantified by the correction factor k, calculated using the following expression:

$$k = \frac{w_{cl} + v}{w_{cl}} = \frac{w}{w_{cl}}, \tag{8}$$

To study the effect of transverse shear deformations on the stiffness of the beams, the following design models were considered: short simply supported and short cantilever beams with a relative beam length $L = 5h$ (where h is the section height) subjected to concentrated and uniformly distributed loads. The selected thin-walled cross-sections include I-shape No. 20 (DSTU 8768:2018 Ukraine) and an annular section with diameter and thickness of 200x2.5 mm. The parametric cross-sections include a rectangular section with a height and width of 200x100 mm and a circular section with a diameter of 200 mm. All beams are made of isotropic material with $G = E/2(1 + v)$.

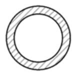
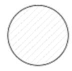
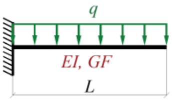
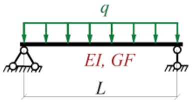
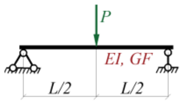
For each case, the values of the correction factor k as ratio of the maximum deflections were computed using formula (8) and presented in Table 1.

The analysis of the obtained correction factor in Table 1 indicates that thin-walled cross-sections are more sensitive to transverse shear deformations than parametric ones. Among these, the I-shaped cross-section is the most susceptible. The design model most sensitive to transverse shear deformations is as follows: a simply supported I-shaped beam under a concentrated load at midspan.

Figure 2 presents a graph illustrating at which number of the I-beam and relative beam length the effect of transverse shear deformations in beams with different design models reaches a value of $k = 1.05$.

From Figure 2, it is evident that the effect of transverse shear deformations must be considered not only for short I-shaped beams but even for relatively "slender" beams. For example, for the I-beam number 10, $k = 1.05$ at $L = 17.9h$. A general trend is observed: the smaller the I-beam No. (the height h), the greater the relative span at which the effect of transverse shear deformations reaches $k = 1.05$.

Table 1 – The correction factor $\boldsymbol{k}$ for the isotropic beams with relative length $\boldsymbol{L = 5h}$
Таблиця 1 – Коефіцієнт уточнення для ізотропної балки з відносною довжиною $\boldsymbol{L = 5h}$

Design model $\boldsymbol{L = 5h}$	 №20 (200mm)	 200x2.5mm	 200x100 mm	 Ø200mm
	1.15	1.08	1.03	1.02
	1.20	1.10	1.04	1.03
	1.48	1.24	1.10	1.07
	1.60	1.30	1.12	1.09

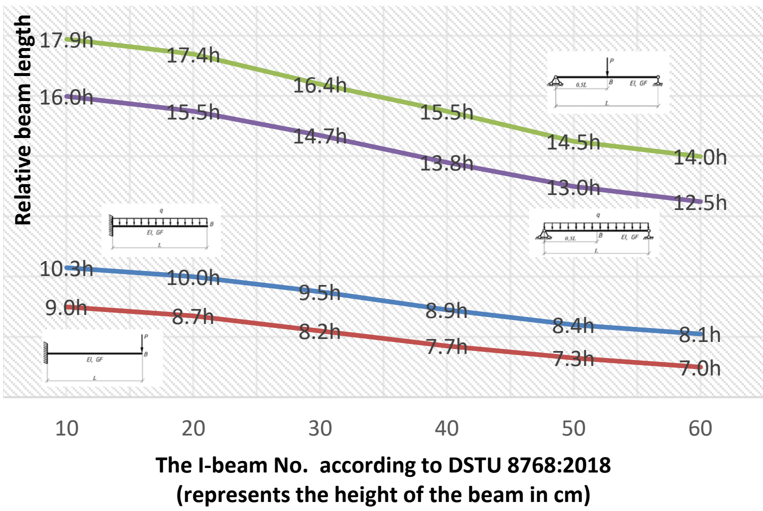


Figure 2 – The effect of transverse shear deformations on I-shaped beams
Рисунок 2 – Вплив деформації поперечного зсуву на двотаврові балки

The reliability of the obtained results, derived from solving equation (1), can be assessed by comparing them with a three-dimensional solution. For the three-dimensional modeling, the SCAD and LIRA software packages are employed [3, 4]. The analysis of short isotropic cantilever beams with a relative length of $L = 5h$ and cross-sections most susceptible to transverse shear deformation (I-shaped and annular cross-sections) is performed (Fig. 3).

In the SCAD and LIRA software packages for the modeling of cantilever beams with I-shaped and annular cross-sections, finite elements FE 41 (arbitrary rectangular FE of shell) were used. The height of the I-shaped cantilever beam was modeled as (200-8.4)=191.6 mm (8.4 mm is the width of the flanges), the diameter of the annular cantilever beam was modeled as (200-2.5)=197.5 mm (2.5 mm is the thickness of the annular section). No offsets for the shells were used. The rigid fixation was modeled by restricting all possible degrees of freedom along the entire height of the corresponding cross-section.

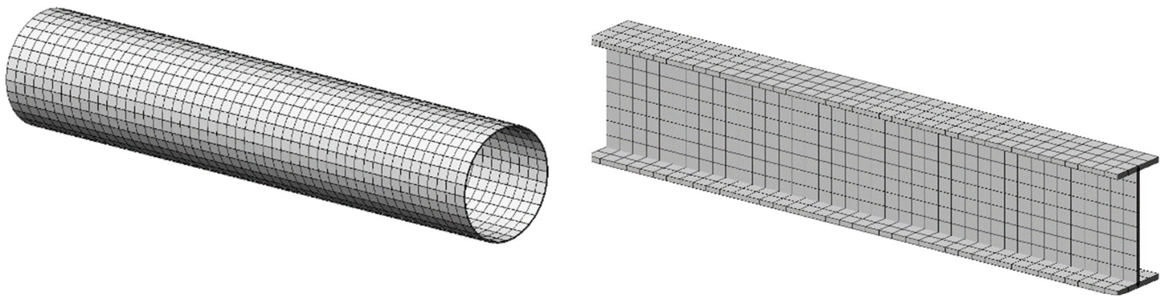


Figure 3 – The finite elements models of beams simulated using the LIRA software
Рисунок 3 – Скінченно-елементні моделі балок, побудованих за допомогою обчислювального комплексу LIRA

The finite element models are illustrated in Figure 3. The loads applied at the free end of the cantilever beam, in the three-dimensional modeling using the SCAD and LIRA software packages, were modeled as shown in Figure 4. The value of the concentrated load is $\mathrm{P} = 10\mathrm{kN}$, and the intensity of the uniformly distributed load is $\mathrm{q} = 10\mathrm{kN/m}$. The modulus of elasticity is $\mathrm{E} = 2.0601 \cdot 10^5\mathrm{MPa}$, material – isotropic.

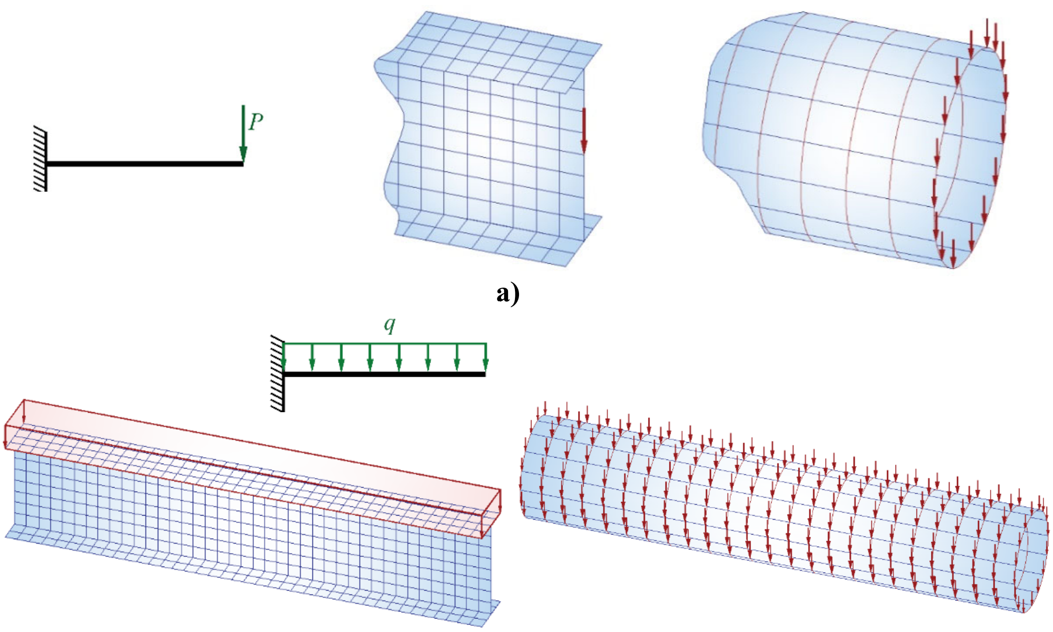


Figure 4 – The three-dimensional finite element modeling of the cantilever beams under the concentrated (a) and uniformly distributed loads (b)
Рисунок 4 – Тривимірне скінченно-елементне моделювання консольних балок під дією зосередженого (a) та рівномірно навантаженого навантаження (b)

The deflection values obtained from the three-dimensional solution were taken for the points as shown in Figure 5.

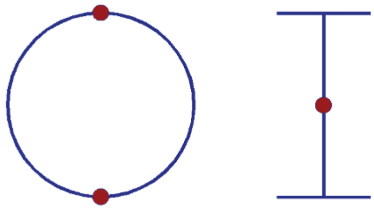
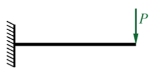
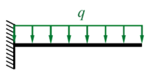


Figure 5 **–** The points at which the deflection values were taken (cross-section at $\boldsymbol{x} = \boldsymbol{L}$ of the cantilever beam)
Рисунок 5 – Точки, в яких було визначено прогин (поперечні перерізи знаходяться на відстані $x = L$)

The discrepancy between analytical calculations based on equation (1) and three-dimensional finite element analysis using the SCAD and LIRA software packages is presented in Table 2 and Table 3.

To assess mesh convergence of the finite element analysis (FEA), the calculation on different finite element meshes was performed. Four types of finite element meshes were considered (Mesh 1, Mesh 2, Mesh 3, Mesh 4). Each subsequent mesh is refined by a factor of two. Thus, mesh convergence graphs of FEA for the deflections at $x = L$ of the cantilever beams $w(L)$ can be plotted.

Table 2 **–** Comparison of the results of the analytical solution and the three-dimensional finite element solution for the I-shaped cantilever beams
Таблиця 2 **–** Порівняння результатів, отриманих на основі аналітичного та тривимірного скінченно-елементного розв'язку для двотаврових консольних балок

 № 20 (200 mm) $\boldsymbol{L = 5h}$	The deflection $\boldsymbol{w(L)}$ in mm obtained from the three-dimensional solution using the SCAD and LIRA			
Design model	Mesh 1 189 nodes/ 160 elements	Mesh 2 697 nodes/ 640 elements	Mesh 3 2673 nodes/ 2560 elements	Mesh 4 10465 nodes/ 10240 elements
	SCAD/LIRA	SCAD/LIRA	SCAD/LIRA	SCAD/LIRA
The deflection w =1.01185 mm obtained by solving equation (1) 	1.00238/ 1.00225	1.01200/ 1.01187	1.01758/ 1.01745	1.02211/ 1.02198
Discrepancy	-0.9%/-1.0%	0%/0%	+0.6%/+0.6%	+1.0%/+1.0%
The deflection w =0.396005 mm obtained by solving equation (1) 	0.389676/ 0.389625	0.392085/ 0.392034	0.392708/ 0.392657	0.392882/ 0.392831
Discrepancy	–1.6%/–1.6%	–1.0%/–1.0%	–0.8%/–0.9%	–0.8%/–0.8%

Graphs of mesh convergence for the deflections $w(L)$ at $x = L$ of the I-shaped cantilever beams are represented in Figure 6.

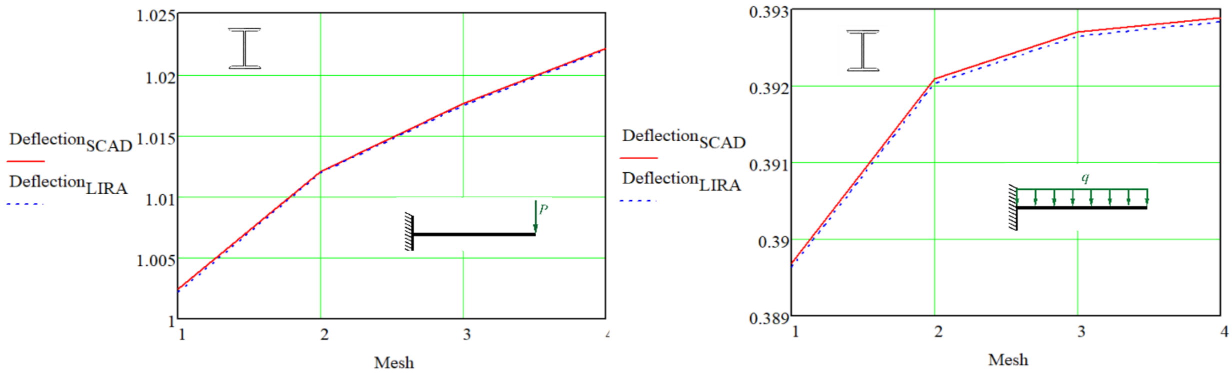


Figure 6 – Mesh convergence graphs for $\boldsymbol{w(L)}$ of the I-shaped cantilever beams obtained from FEA

Рисунок 6 – Графіки практичної збіжності МСЕ по прогину $w(L)$ для двотаврових консольних балок

Table 3 **–** Comparison of the results of the analytical solution and the three-dimensional solution for the cantilevers with annular cross-sections

Таблиця 3 – Порівняння результатів, отриманих на основі аналітичного та тривимірного скінченно-елементного розв'язку для консольних балок з кільцевим поперечним перерізом

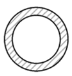
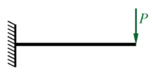
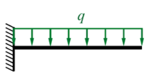
 200x2.5 mm $\boldsymbol{L = 5h}$	The deflection $\boldsymbol{w(L)}$ in mm obtained from the three-dimensional solution using the SCAD and LIRA			
Design model	Mesh 1 104 nodes/ 96 elements	Mesh 2 400 nodes, 384 elements	Mesh 3 1568 nodes/ 1536 elements	Mesh 3 6208 nodes/ 6144 elements
	SCAD/LIRA	SCAD/LIRA	SCAD/LIRA	SCAD/LIRA
The deflection w =2.34839 mm obtained by solving equation (1) 	2.51557/ 2.51493	2.36715/ 2.36693	2.34372/ 2.34346	2.34316/ 2.34299
Discrepancy	+7.1%/+7.1%	+0.8%/+0.8%	-0.2%/-0.2%	-0.2%/-0.2%
The deflection w =0.883505 mm obtained by solving equation (1) 	0.954521/ 0.954221	0.904609/ 0.904507	0.890042/ 0.889943	0.885946/ 0.885850
Discrepancy	+8.0%/+8.0%	+2.4%/+2.4%	+0.7%/+0.7%	+0.3%/+0.3%

Figure 6 illustrates that the three-dimensional finite element model of the I-shaped cantilever beam, under uniformly distributed loads, indicates mesh convergence.

However, the situation is unlike that of the cantilever beam under a concentrated load. This may be due to the idealization of the concentrated load, which, in reality, can lead to significant local effects that are not always accurately accounted for in finite element modeling.

Graphs of mesh convergence for the deflections $w(L)$ at $x = L$ of the annular cantilever beams are represented in Figure 7.

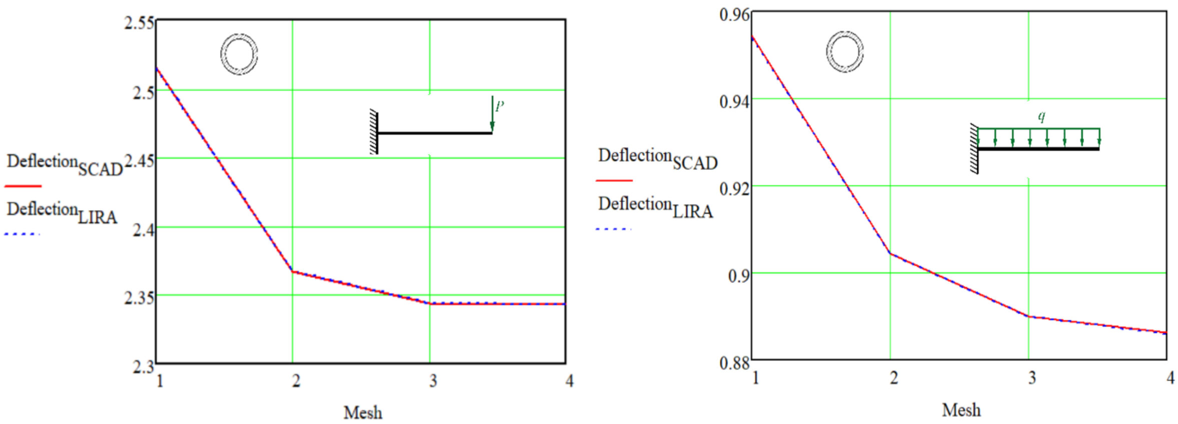


Figure 7 – Convergence graphs for $\boldsymbol{w(L)}$ of the annular cantilever beams obtained from FEA
Рисунок 6 – Графіки практичної збіжності МСЕ по прогину $w(L)$ для консольних балок з кільцевим поперечним перерізом

Figure 7 illustrates that the three-dimensional finite element model of the annular cantilever beam, under both concentrated and uniformly distributed loads, indicates mesh convergence. The negative effect of the concentrated load on the three-dimensional finite element model was avoided by modeling the concentrated load as a system of equivalent forces applied at the nodes of the corresponding section.

Table 2 and 3 demonstrate a high degree of accuracy between the analytical solution and the three-dimensional finite element solution. The three-dimensional solution, obtained using the SCAD and LIRA-FEM software packages, is a numerical approach that models the actual support conditions and load transfer to the beam in a more detailed and realistic manner.

Conclusions

1. This study analyzed beams with different design models and cross-sections: thin-walled (I-shaped, annular) and parametric (rectangular, circular). For each case, the measure of correction factor k (the ratio of the maximum deflection calculated by considering both pure bending and transverse shear deformations to the maximum deflection calculated by considering only the pure bending deformations) was calculated that indicates the effect of transverse shear deformations on the stiffness of the beams. The design model most susceptible to transverse shear deformations was identified. It appeared to be a simply supported I-shaped beam under a concentrated load at a midspan.

2. Considering transverse shear deformations is necessary not only for short I-shaped beams with a relative length $L \leq 5h$ (h – beam height) but even for relatively long I-shaped beams. The smaller the I-beam No. (the height h), the greater the relative span at which the influence of transverse shear deformations reaches $k = 1.05$.

3. Comparison of calculations based on the differential equation, which accounts for both pure bending and transverse shear deformations, with three-dimensional finite element analysis using the SCAD and LIRA

software packages demonstrates a high degree of accuracy between analytical and numerical solutions. The I-shaped and annular cantilever beams as the most susceptible to transverse shear deformations cross-sections were analyzed using FEA. Mesh convergence graphs were plotted for the finite element solutions, and all design models of the cantilever beams indicated a convergence except for the I-shaped cantilever beam under a concentrated load. This may be due to the idealization of the concentrated load which, in reality, can lead to significant local effects that are not always accurately accounted for in finite element modeling.

4. The obtained results have practical significance for structural engineers, particularly in analyzing short beam deformations during the design of beams with different cross-sections. The research findings can be incorporated into the educational process for the "Strength of Materials" course.

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АНАЛІЗ ВПЛИВУ ДЕФОРМАЦІЙ ПОПЕРЕЧНОГО ЗСУВУ НА ЖОРСТКІСТЬ КОРОТКИХ БАЛОК

Ільченко Ірина Олексіївна, кандидат технічних наук, доцент, Національний транспортный університет, Київ, Україна, доцент кафедри опору матеріалів та машинознавства, e-mail: budivmeh@gmail.com, тел. +380930077702, Україна, 01010, м. Київ, вул. Омеляновича-Павленка, 1, кімн. 113, https://orcid.org/0000-0002-9355-5632

Горбунович Ірина Валентинівна, кандидат технічних наук, доцент, Національний транспортный університет, Київ, Україна, доцент кафедри вищої математики, e-mail: irina.gorbunovich@gmail.com, тел. +380676001177, Україна, 01103, м. Київ, вул Бойчука, буд. 42, к.514, orcid.org 0000-0002-6859-0663.

Анотація. У статті вивчаються деформації коротких пружних балок різного поперечного перерізу, що виникають під дією зовнішнього зосередженого та рівномірно розподіленого навантаження. Досліджується вплив деформацій поперечного зсуву на жорсткість балок.

Як математична модель задачі пропонується диференціальне рівняння другого порядку, побудоване з урахуванням деформацій чистого згинання й поперечного зсуву [1]. Граничні умови задано для шарнірно-опертих і консольних балок.

Крайова задача розв'язується аналітично і чисельним методом скінченних елементів (МСЕ), який дозволяє керувати різним рівнем точності.

Знайдено максимальні прогини балок з різними типами перерізів (прямокутним, двотавровим, круглим та кільцевим). Для знаходження прогинів в балках застосовано теорію балок С.П.Тимошенка [2].

Проте сучасне проєктування складних об'єктів неможливе без використання систем автоматизованого проєктування (САПР), які дозволяють створювати більш точні моделі конструкцій та аналізувати їх поведінку в умовах, наближених до реальних. За допомогою САПР (з вбудованими інструментами скінченно-елементного аналізу) можна досліджувати поведінку та виявляти особливості деформацій як окремих елементів, так і об'єкта в цілому [3].

Тривимірне скінченно-елементне моделювання консольних балок виконано на основі обчислювальних комплексів SCAD та LIRA. Побудовано графіки практичної збіжності по прогинам для тонкостінних консольних балок. Порівняння аналітичного і чисельного розв'язків крайової задачі показало високий рівень їх збіжності, що свідчить про точність і ефективність застосованих підходів. Отримані результати мають практичне значення для інженерів-проектувальників, зокрема в контексті аналізу деформацій коротких балок у процесі розробки конструкцій з різними геометричними перерізами. Результати досліджень також варто використовувати в навчальному процесі під час викладання курсу «Опір матеріалів».

Ключові слова: шарнірно-оперта балка, консольна балка, тонкостінні поперечні перерізи, параметричні поперечні перерізи, жорсткість балки, деформації поперечного зсуву, скінченно-елементний обчислювальний комплекс, обчислювальний комплекс LIRA, обчислювальний комплекс SCAD.

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